

Inequalities for ultraspherical polynomials. Proof of a conjecture of I. Raşa

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Abstract

A recent conjecture by I. Raşa asserts that the sum of the squared Bernstein basis polynomials is a convex function in $[0, 1]$. This conjecture turns out to be equivalent to a certain upper pointwise estimate of the ratio $P'_n(x)/P_n(x)$ for $x \geq 1$, where P_n is the n -th Legendre polynomial. Here, we prove both upper and lower pointwise estimates for the ratios $(P_n^{(\lambda)}(x))'/P_n^{(\lambda)}(x)$, $x \geq 1$, where $P_n^{(\lambda)}$ is the n -th ultraspherical polynomial. In particular, we validate Raşa's conjecture.

Key words: Bernstein polynomials, Legendre polynomials, ultraspherical polynomials

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1 Introduction and statement of the results

The classical Bernstein operator

$$B_n(f; t) = \sum_{k=0}^n b_{n,k}(t) f\left(\frac{k}{n}\right),$$

where the basis polynomials $b_{n,k}$ are given by

$$b_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}, \quad k = 0, 1, \dots, n,$$

plays an important role in Approximation Theory, and has been an object of intensive study throughout the years, see [1, Chapter 10].

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Recently, Ioan Raşa [7] (see [4] for a more accessible source) formulated the following conjecture about the sum of the squared Bernstein basis polynomials:

Conjecture 1 *The function*

$$F_n(t) := \sum_{k=0}^n b_{n,k}^2(t)$$

is convex in $[0, 1]$.

As a matter of fact, Conjecture 1 is the strongest amongst three conjectures proposed by Raşa in [7]; the other two are that $F_n(t)$ attains its minimum in $[0, 1]$ at $t = 1/2$, and that $F_n(t)$ is monotone decreasing in $[0, 1/2]$ and monotone increasing in $[1/2, 1]$ (see [4, Conjectures 3.4 and 3.3]). These weaker conjectures have been validated by Thorsten Neuschel [6] (see also [4, Lemma 3.5]). The function F_n appears as a factor in an upper estimate of the degree of non-multiplicativity of Bernstein operators, see [4].

In Section 2 we prove the following equivalence:

Theorem 2 *Conjecture 1 is equivalent to the inequality*

$$\frac{P'_n(x)}{P_n(x)} \leq \frac{2n^2}{x + (2n - 1)\sqrt{x^2 - 1}}, \quad x \geq 1, \quad (1.1)$$

where P_n is the n -th Legendre polynomial.

As is well-known, the Legendre polynomial P_n belongs to the family of ultraspherical polynomials $P_n^{(\lambda)}$, $\lambda > -1/2$, which are orthogonal in $[-1, 1]$ with respect to the weight function $w_\lambda(x) = (1 - x^2)^{\lambda-1/2}$ (P_n corresponds to the case $\lambda = 1/2$). This is our motivation instead of proving inequality (1.1) to obtain upper (but also lower) pointwise bounds for the ratios $(P_n^{(\lambda)}(x))' / P_n^{(\lambda)}(x)$, $x \geq 1$. For the sake of brevity, hereafter we skip the superscript (λ) and write

$$p_n(x) := P_n^{(\lambda)}(x), \quad u_n(x) := \frac{p'_n(x)}{p_n(x)}.$$

In Section 3 we prove

Theorem 3 *Let $n \in \mathbb{N}$ and $\lambda > -1/2$. Then*

$$u_n(x) \geq \frac{n(n + 2\lambda)}{(2\lambda + 1)x + (n - 1)\sqrt{x^2 - 1}}, \quad x \geq 1. \quad (1.2)$$

Moreover, if $\lambda \in [0, 1]$, then

$$u_n(x) \leq \frac{n^2(n + \lambda)}{\lambda(n + 1)x + (n^2 - \lambda)\sqrt{x^2 - 1}}, \quad x \geq 1. \quad (1.3)$$

Specialized to the case $\lambda = 1/2$, inequality (1.3) becomes

$$\frac{P'_n(x)}{P_n(x)} \leq \frac{n^2(2n+1)}{(n+1)x + (2n^2-1)\sqrt{x^2-1}}, \quad x \geq 1,$$

which is easily seen to be slightly stronger than (1.1). Consequently, we have

Corollary 4 *Conjecture 1 is true.*

Yet another theorem of the same nature is

Theorem 5 *Let $n \in \mathbb{N}$ and $\lambda > -1/2$. Then*

$$n\left(\frac{1}{x} + \frac{n-1}{2(n+\lambda-1)x^3}\right) \leq u_n(x) \leq n\left(\frac{1}{x} + \frac{n-1}{(2\lambda+1)x^3}\right), \quad x \geq 1. \quad (1.4)$$

The proof of Theorem 5 is given in Section 4. In the final section we compare Theorems 3 and 5 and discuss their sharpness.

2 Equivalent formulation of Conjecture 1: Proof of Theorem 2

Clearly,

$$F_n(t) = \sum_{k=0}^n \binom{n}{k}^2 t^{2k} (1-t)^{2n-2k}$$

is a polynomial of degree $2n$, and $F_n(t) = F_n(1-t)$. Therefore, Conjecture 1 is equivalent to the inequality

$$F''_n(t) \geq 0, \quad t \in [0, 1/2]. \quad (2.1)$$

We shall express F_n through the Legendre polynomial P_n . By Rodrigues' formula and the Leibnitz rule we have

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \{(x^2-1)^n\} = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k}^2 (x-1)^k (x+1)^{n-k}. \quad (2.2)$$

Let us set

$$x = x(t) := \frac{1}{2} \left(1 - 2t + \frac{1}{1-2t} \right), \quad t \in [0, 1/2),$$

then x traces the interval $[1, \infty)$, and

$$x-1 = \frac{2t^2}{1-2t}, \quad x+1 = \frac{2(1-t)^2}{1-2t}, \quad 1-2t = x - \sqrt{x^2-1}.$$

Replacement in (2.2) yields

$$P_n(x) = \frac{1}{(1-2t)^n} \sum_{k=0}^n \binom{n}{k}^2 t^{2k} (1-t)^{2n-2k} = (1-2t)^{-n} F_n(t),$$

hence

$$F_n(t) = \left(x - \sqrt{x^2 - 1}\right)^n P_n(x). \quad (2.3)$$

Taking into account that

$$x'(t) = \frac{1}{(1-2t)^2} - 1 = \frac{4t(1-t)}{(1-2t)^2} = \frac{2\sqrt{x^2-1}}{x - \sqrt{x^2-1}},$$

we differentiate (2.3) to obtain consecutively

$$\begin{aligned} F'_n(t) &= \frac{d}{dx} \left\{ (x - \sqrt{x^2-1})^n P_n(x) \right\} x'(t) \\ &= 2(x - \sqrt{x^2-1})^{n-1} \left(\sqrt{x^2-1} P'_n(x) - n P_n(x) \right) \end{aligned} \quad (2.4)$$

(this formula has been already obtained by Neuschel [6]),

$$\begin{aligned} F''_n(t) &= 2 \frac{d}{dx} \left\{ (x - \sqrt{x^2-1})^{n-1} \left(\sqrt{x^2-1} P'_n(x) - n P_n(x) \right) \right\} x'(t) \\ &= 4(x - \sqrt{x^2-1})^{n-2} \\ &\quad \times \left((x^2-1) P''_n(x) + \left(x - (2n-1)\sqrt{x^2-1} \right) P'_n(x) + n(n-1) P_n(x) \right). \end{aligned}$$

On using the differential equation

$$(1-x^2)P''_n(x) - 2x P'_n(x) + n(n+1)P_n(x) = 0,$$

we replace $(x^2-1)P''_n(x)$ to finally obtain

$$F''_n(t) = 4(x - \sqrt{x^2-1})^{n-2} \left[2n^2 P_n(x) - \left(x + (2n-1)\sqrt{x^2-1} \right) P'_n(x) \right]. \quad (2.5)$$

Since $x - \sqrt{x^2-1} > 0$ for $x \geq 1$, it follows that (2.1) is fulfilled exactly when the term in the square brackets in (2.5) is non-negative. The latter is equivalent to (1.1), as $P_n(x) > 0$ for $x \geq 1$. Theorem 2 is proved.

3 Proof of Theorem 3

For easy reference, we collect in a lemma some well-known properties of ultraspherical polynomials, see, e.g., [8, Chapter 4.7].

Lemma 6 For $\lambda \neq 0$, $p_n = P_n^{(\lambda)}$ satisfies the following properties:

- (i) $(1 - x^2)p_n''(x) - (2\lambda + 1)x p_n'(x) + n(n + 2\lambda)p_n(x) = 0;$
- (ii) $p_{n+1}'(x) = (n + 2\lambda)p_n(x) + x p_n'(x);$
- (iii) $p_{n+1}(x) = \frac{1}{n + 1} \left((n + 2\lambda)x p_n(x) + (x^2 - 1)p_n'(x) \right);$
- (iv) $n p_n(x) = x p_n'(x) - p_{n-1}'(x);$
- (v) $(n + 1)p_{n+1}(x) = 2(n + \lambda)x p_n(x) - (n + 2\lambda - 1)p_{n-1}(x), \quad n \geq 1.$

Set

$$u_n(x) := \frac{p_n'(x)}{p_n(x)}, \quad n \in \mathbb{N},$$

then obviously $u_n(x)$ is positive and strictly monotone decreasing in $[1, \infty)$. On using Lemma 6 (i), we find

$$u_n(1) = \frac{n(n + 2\lambda)}{2\lambda + 1}. \quad (3.1)$$

The proof of Theorem 3 goes by induction. For the induction transition from n to $n + 1$, we make use of Lemma 6 (ii) and (iii) to obtain

$$u_{n+1}(x) = (n + 1) \frac{n + 2\lambda + x u_n(x)}{(n + 2\lambda)x + (x^2 - 1) u_n(x)}. \quad (3.2)$$

We observe that the function $\varphi(t) = \frac{a+xt}{ax+(x^2-1)t}$ is continuous and strictly monotone increasing in $(0, \infty)$ whenever $a > 0$ and $x \geq 1$.

Let us prove first inequality (1.2). Clearly $u_1(x) = 1/x$ satisfies (1.2) with equality for every $x \geq 1$. Suppose that, for some $n \in \mathbb{N}$,

$$u_n(x) \geq \frac{n(n + 2\lambda)}{(2\lambda + 1)x + (n - 1)\sqrt{x^2 - 1}} =: t_n(x), \quad x \geq 1.$$

Then, by (3.2),

$$u_{n+1}(x) \geq (n + 1) \frac{n + 2\lambda + x t_n(x)}{(n + 2\lambda)x + (x^2 - 1) t_n(x)},$$

and the induction step will be performed once we show that

$$(n + 1) \frac{n + 2\lambda + x t_n(x)}{(n + 2\lambda)x + (x^2 - 1) t_n(x)} \geq t_{n+1}(x), \quad x \geq 1.$$

A straightforward calculation shows that the latter inequality is equivalent to the obvious inequality

$$2(\lambda + 1)n\sqrt{x^2 - 1}(x - \sqrt{x^2 - 1}) \geq 0, \quad x \geq 1,$$

and this accomplishes the induction proof of (1.2).

Inequality (1.3) can be proved by induction along the same lines as (1.2). However, we would like to provide some clue about the way we deduced this inequality.

We seek for which non-negative $c_n = c_n(\lambda)$ the inequality

$$u_n(x) \leq \frac{n^2}{c_n x + (n - c_n)\sqrt{x^2 - 1}} =: \tau(n, c_n, x), \quad x \geq 1, \quad (3.3)$$

holds true. As is easy to see, the larger c_n , the better (i.e., smaller) the upper bound $\tau(n, c_n, x)$ in (3.3). However, c_n cannot be arbitrarily large, for, according to (3.1), in order that (3.3) is true for $x = 1$, there must hold $n^2/c_n \geq n(n + 2\lambda)/(2\lambda + 1)$. Hence, a natural restriction for c_n is

$$0 \leq c_n \leq \frac{(2\lambda + 1)n}{n + 2\lambda}.$$

Assume that (3.3) holds true for some $n \in \mathbb{N}$. By (3.2), we have

$$u_{n+1}(x) \leq (n + 1) \frac{n + 2\lambda + x \tau(n, c_n, x)}{(n + 2\lambda)x + (x^2 - 1) \tau(n, c_n, x)}, \quad x \geq 1.$$

The induction step will be done if we manage to show that

$$(n + 1) \frac{n + 2\lambda + x \tau(n, c_n, x)}{(n + 2\lambda)x + (x^2 - 1) \tau(n, c_n, x)} \leq \tau(n + 1, c_{n+1}, x), \quad x \geq 1. \quad (3.4)$$

At this point we assume that the sequence $\{c_n\}_{n=1}^\infty$ is non-increasing. Then (3.4) will be a consequence of the inequality

$$(n + 1) \frac{n + 2\lambda + x \tau(n, c_n, x)}{(n + 2\lambda)x + (x^2 - 1) \tau(n, c_n, x)} \leq \tau(n + 1, c_n, x), \quad x \geq 1, \quad (3.5)$$

since $\tau(n, c, x)$ is a decreasing function of c . Now we check for which c_n the inequality (3.5) holds true. Inequality (3.5) is equivalent (for brevity, here we write c instead of c_n) to

$$\frac{[n^2 + c(n + 2\lambda)]x + (n - c)(n + 2\lambda)\sqrt{x^2 - 1}}{c(n + 2\lambda)x^2 + (n - c)(n + 2\lambda)x\sqrt{x^2 - 1} + n^2(x^2 - 1)} \leq \frac{n + 1}{cx + (n + 1 - c)\sqrt{x^2 - 1}}.$$

Since $x \geq 1$, both denominators are positive, and after simplification and cancelation of the positive factor $x - \sqrt{x^2 - 1}$, the above inequality is reduced to the inequality

$$c[2\lambda(n+1)+n-c(n+2\lambda)]x - [(2n+1)(n+2\lambda)c - (n+2\lambda)c^2 - 2\lambda n(n+1)]\sqrt{x^2-1} \geq 0.$$

For the last inequality to be true for every $x \geq 1$, the coefficient of x must be non-negative and greater than or equal to the coefficient of $\sqrt{x^2 - 1}$, i.e., there must hold

$$c[2\lambda(n+1)+n-c(n+2\lambda)] \geq \max\{0, (2n+1)(n+2\lambda)c - (n+2\lambda)c^2 - 2\lambda n(n+1)\}.$$

The latter is equivalent to

$$0 \leq c \leq \min \left\{ \frac{2\lambda(n+1)+n}{n+2\lambda}, \frac{\lambda(n+1)}{n+\lambda} \right\}. \quad (3.6)$$

Clearly, if $-1/2 < \lambda < 0$, then (3.6) to has no solution. On the other hand, if $0 \leq \lambda \leq 1$, then

$$c = c_n = \frac{\lambda(n+1)}{n+\lambda}$$

is a solution of (3.6), and the sequence $\{c_n\}_{n=1}^{\infty}$ is non-increasing, in accordance with our assumption.

Performing our reasoning backward, we see that if $c_n = \frac{\lambda(n+1)}{n+\lambda}$, $0 \leq \lambda \leq 1$, then $u_n(x) \leq \tau(n, c_n, x)$ for every $x \geq 1$ implies $u_{n+1}(x) \leq \tau(n+1, c_{n+1}, x)$ for every $x \geq 1$, i.e., the induction step is done. Notice that for this choice of c_n we have

$$\tau(n, c_n, x) = \frac{n^2(n+\lambda)}{\lambda(n+1)x + (n^2 - \lambda)\sqrt{x^2 - 1}},$$

therefore (3.3) is in fact inequality (1.3). It remains to verify (1.3) for $n = 1$, i.e.,

$$\frac{1}{x} \leq \frac{1+\lambda}{2\lambda x + (1-\lambda)\sqrt{x^2-1}}, \quad x \geq 1.$$

The latter inequality is equivalent to $(1-\lambda)(x - \sqrt{x^2-1}) \geq 0$, hence is true. The proof of Theorem 3 is complete.

4 Proof of Theorem 5

On using Lemma 6 (iv) we obtain

$$u_n(x) = \frac{1}{x} \left(n + \frac{p'_{n-1}(x)}{p_n(x)} \right). \quad (4.1)$$

Let $\{x_k\}_{k=1}^n$ be the zeros of $p_n = P_n^{(\lambda)}$, they form a symmetrical set with respect to the origin. Invoking again Lemma 6 (iv) we get

$$\frac{p'_{n-1}(x_k)}{p'_n(x_k)} = x_k, \quad k = 1, \dots, n.$$

By Lagrange's interpolation formula and the symmetry of the set $\{x_k\}_{k=1}^n$ we obtain

$$\begin{aligned} \frac{p'_{n-1}(x)}{p_n(x)} &= \sum_{k=1}^n \frac{p'_{n-1}(x_k)}{p'_n(x_k)} \cdot \frac{1}{x - x_k} = \sum_{k=1}^n \frac{x_k}{x - x_k} \\ &= \frac{1}{2} \sum_{k=1}^n \left(\frac{x_k}{x - x_k} - \frac{x_k}{x + x_k} \right) = \sum_{k=1}^n \frac{x_k^2}{x^2 - x_k^2}. \end{aligned} \quad (4.2)$$

Hence,

$$\psi(x) := \frac{x^2 p'_{n-1}(x)}{p_n(x)} = \sum_{k=1}^n \frac{x_k^2}{1 - (x_k/x)^2},$$

and

$$\psi'(x) = -\frac{2}{x^3} \sum_{k=1}^n \frac{x_k^4}{(1 - (x_k/x)^2)^2} < 0 \quad \text{for } x \geq 1.$$

We observe that ψ is a monotone decreasing function in $[1, \infty)$, therefore $\psi(1) \geq \psi(x) \geq \lim_{x \rightarrow \infty} \psi(x)$ therein. Lemma 6 (iv) and (3.1) imply

$$\psi(1) = u_n(1) - n = \frac{n(n-1)}{2\lambda+1}. \quad (4.3)$$

On the other hand, we have

$$\lim_{x \rightarrow \infty} \psi(x) = (n-1) \frac{a_{n-1}}{a_n}$$

with $a_m = a_m(\lambda)$ being the leading coefficient of p_m . From $a_0 = 1$, $a_1 = 2\lambda$ and Lemma 6 (v) we infer

$$a_m = \frac{2^m \lambda(\lambda+1) \cdots (\lambda+m-1)}{m!}, \quad m \in \mathbb{N},$$

whence

$$\lim_{x \rightarrow \infty} \psi(x) = \frac{n(n-1)}{2(n+\lambda-1)}. \quad (4.4)$$

Thus,

$$\frac{n(n-1)}{2(n+\lambda-1)x^2} \leq \frac{p'_{n-1}(x)}{p_n(x)} \leq \frac{n(n-1)}{(2\lambda+1)x^2}, \quad x \in [1, \infty).$$

Theorem 5 follows from substituting these bounds in (4.1).

5 Final remarks

The bounds for $u_n(x) = p'_n(x)/p_n(x)$ provided by Theorems 3 and 5 are sharp as $x \rightarrow \infty$ in the sense that they preserve the property $\lim_{x \rightarrow \infty} x u_n(x) = n$. The lower bound in Theorem 3 and the upper bound in Theorem 5 are also sharp for $x = 1$, see (3.1). However, except for some neighborhoods of $x = 1$, the latter bounds are inferior to their counterparts given in Theorem 5 and Theorem 3, respectively.

Unfortunately, the upper bound in Theorem 3 was only proven for $0 \leq \lambda \leq 1$. This restriction on λ is not a proof defect, as for negative λ the right-hand side of (1.3) is negative at $x = 1$ while if $\lambda > 1$ then (1.3) fails for $n = 1$.

As is well-known (see, e.g., [8, Theorem 6.21.1]), the squared k -th zero $x_k^2 = x_k^2(\lambda)$ of $p_n = P_n^{(\lambda)}$ is a monotone decreasing function of λ , ($k = 1, \dots, n$). By (4.2) we deduce that

$$\frac{p'_{n-1}(x)}{p_n(x)} = \sum_{k=1}^n \frac{1}{x^2/x_k^2 - 1}$$

is also a monotone decreasing function of λ for $x \geq 1$, and so is $u_n(x)$, by virtue of (4.1). Therefore, the upper bound for $u_n(x)$ given by (1.3) for $\lambda = 1$ is also upper bound for $u_n(x)$ whenever $\lambda \geq 1$, i.e.,

$$u_n(x) \leq \frac{n^2}{x + (n-1)\sqrt{x^2-1}} \quad \text{for every } x \geq 1 \quad \text{and } \lambda \geq 1. \quad (5.1)$$

For particular $\lambda \geq 1$, n and $x > 1$ (5.1) may provide better upper bound for $u_n(x)$ than the one given by Theorem 5.

Of course, in the cases $\lambda = 0$ and $\lambda = 1$ one can obtain explicit formulae for $u_n(x)$ exploiting the representation of the Chebyshev polynomials of the first and second kind (see, e.g., [5, Eqns. (1.49) and (1.52)])

$$T_n(x) = \frac{1}{2} \left((x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n \right),$$

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}} \left((x + \sqrt{x^2-1})^{n+1} - (x - \sqrt{x^2-1})^{n+1} \right).$$

Let us finally point out that for fixed $n \in \mathbb{N}$ and $x \geq 1$ both the upper and the lower bound in Theorem 5 as well as the lower bound in Theorem 3 are asymptotically sharp as $\lambda \rightarrow \infty$. Indeed, the extreme zeros $x_1(\lambda) = -x_n(\lambda)$ of $P_n^{(\lambda)}$ satisfy (see, e.g. [3] or [2])

$$x_n^2(\lambda) = x_1^2(\lambda) \leq \frac{(n-1)(n+2\lambda+1)}{(n+\lambda)^2} \rightarrow 0 \quad \text{as } \lambda \rightarrow \infty,$$

whence

$$\lim_{\lambda \rightarrow \infty} \frac{P_n^{(\lambda)}(x)}{a_n(\lambda)} = x^n,$$

and consequently

$$\lim_{\lambda \rightarrow \infty} u_n(x) = \frac{n}{x}.$$

Clearly, the bounds for u_n in Theorem 5 as well as the lower bound in Theorem 3 enjoy the same limit.

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